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Constitutive modeling of discontinuities by means of discrete and continuum approximations and damage models

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Abstract

In this paper the mathematical modeling of discontinuities using the discrete approximation and the continuum approximation with weak discontinuities is presented. First, the kinematics of discontinuities is discussed, then two constitutive models based on the continuum damage mechanics theory are developed. The first model is an isotropic damage model and is used in the discrete approximation. The second model is an anisotropic damage model and is used in the continuum approximation. These models are characterized for weighing the mode of failure in the failure criterion. An energy analysis is proposed to establish the equations that relate the parameters of both constitutive models; the fulfillment of the involved equations guarantee that both models are energetically equivalent. It is concluded that the proposed models are suitable to reproduce the constitutive behavior of discontinuities.

Keywords: Damage models; Discontinuities; Cracks; Strain localization; Discrete approximation; Continuum approximation

1. Introduction

The failure process in materials and structures can be associated to the formation of single cracks, cracks bands and shear bands. Many models can be found in the literature; for example, in the eighties the effort was on the development of the discrete crack model (Wawrzynek and Ingraffea, 1991) and the smeared crack model (Isenberg, 1993). In the nineties, it was developed a type of finite element model capable of modeling localized damage without remeshing, resulted in the embedded discontinuities approximations (for example: Belytschko et al., 1988; Dvorkin and Assanelli, 1991; Simo et al., 1993; Lotfi and Shing, 1995; Oliver, 1996; Armero and Garikipati, 1996; Sluys and Berends, 1998; Tano et al., 1998; de Borst et al.,

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2001); some of these models does not work out satisfactorily (Jirásek, 2000), showing the need for improvements in some specific topics (Fernández, 2002).

Two types of approximation for embedded discontinuities can be identified: discrete and continuum. In the discrete approximation of discontinuities it is considered that the body stops being continuous when the discontinuity (crack) is formed; the constitutive behavior of the discontinuity is modeled by "displacement jump-traction" relationships (e.g. Dvorkin and Assanelli, 1991; Lotfi and Shing, 1995). In the continuum approximation of discontinuities it is considered that the body remains continuous after the discontinuity (strain localization zone) is formed; the constitutive behavior of the strain localization zone is modeled by a standard continuum type of constitutive equations ("strain–stress" relationships). The continuum approximation of discontinuities is divided in two groups: weak discontinuities (e.g. Belytschko et al., 1988; Sluys and Berends, 1998) and strong discontinuities (e.g. Simo et al., 1993; Oliver, 1996). If the strain localization band width k is very small ($k \rightarrow 0$), a strong discontinuity is considered. If the strain localization band width is finite ($k \gg 0$), a weak discontinuity is considered. Must of the works in the technical literature belong to the discrete approximation and the continuum approximation with strong discontinuities.

The discrete and the continuum approximations actually represent real physical phenomena. The formation and propagation of single cracks (discrete approximation) and shear and crack bands (continuum approximation) have been identified and studied in many materials, such as concrete, steel, soils and rocks. In the case of steel, it has been observed the formation of both types of discontinuities, singles cracks and shear bands. In the case of concrete, the failure process initiates with the formation of microcracks uniformly distributed in bands (strain localization zones), which have an approximate width of three times the maximum aggregate size (Bažant and Oh, 1983).

This paper studies the mathematical modeling of discontinuities. This work is part of an ongoing project on the numerical modeling of discontinuities by the finite element method. The project is composed of five parts: (1) definition of kinematics of discontinuities, (2) variational formulation, (3) constitutive models for discontinuities, (4) implementation in the finite element method and (5) numerical simulation of discontinuities. This paper focuses on the definition of kinematics and the constitutive modeling of discontinuities.

In this paper two different approximations are covered: discrete approximation and continuum approximation with weak discontinuities. First, the kinematics of discontinuities is presented, which defines the displacement field and the strain tensor. Then, to define the constitutive behavior of discontinuities by discrete approximation an isotropic damage model is proposed. For weak discontinuities, due to the limitations of the isotropic damage model for this type of approximation, an anisotropic damage model is developed. Finally, an energy analysis is presented to establish a relationship between both models. It is concluded that the proposed models are suitable to reproduce the constitutive behavior of discontinuities and that they are energetically equivalent if the equations relating the internal variables are fulfilled.

2. Kinematics of a medium with discontinuities

The kinematics of bodies with discontinuities can be established using two different approximations: strong discontinuities or weak discontinuities. In the first, a discontinuous displacement field is defined and it belongs to it the discrete approximation and the continuum approximation with strong discontinuities. In the second, the displacement field is continuous, but the strains are discontinuous at the localization zone boundaries; the continuum approximation with weak discontinuities belongs to this approximation.

In the following two subsections, a description of the kinematics of bodies with discontinuities is presented, defining the displacement field and the strain tensor associated to the strong and weak discontinuity approximations.

2.1. Strong discontinuities

Consider a solid and homogenous body (Fig. 1) whose material points are labeled by the global coordinate system $\mathbf{x}$ (x, y). The body has a domain Ω and a boundary Γ $(\Gamma = \partial\Omega)$. This body has a discontinuity (crack) at S, which is contained in the subdomain Ω_h; the vector $\mathbf{n}$, normal to the discontinuity, defines a local coordinate system $\hat{\mathbf{n}}$ (n, t). The domain is divided into three subdomains: $\Omega = \Omega^- \cup \Omega_h \cup \Omega^+$; the subdomain Ω^+ is that located in the direction of $\mathbf{n}$. The lines S_h^- and S_h^+ bound the subdomain Ω_h, such that: $S_h^- = \partial\Omega_h^- \cap \partial\Omega^-$ and $S_h^+ = \partial\Omega_h^+ \cap \partial\Omega^+$, and are separated by a distance h.

The displacement field $\mathbf{u}(\mathbf{x}, \hat{t})$ and the velocity $\dot{\mathbf{u}}(\mathbf{x}, \hat{t})$ are defined as

$$\mathbf{u}(\mathbf{x}, \hat{t}) = \bar{\mathbf{u}}(\mathbf{x}, \hat{t}) + H_S(\mathbf{x})[[\mathbf{u}]](\mathbf{x}, \hat{t}) \tag{1}$$

$$\dot{\mathbf{u}}(\mathbf{x}, \hat{t}) = \frac{\partial \mathbf{u}(\mathbf{x}, \hat{t})}{\partial \hat{t}} = \dot{\bar{\mathbf{u}}}(\mathbf{x}, \hat{t}) + H_S(\mathbf{x})[[\dot{\mathbf{u}}]](\mathbf{x}, \hat{t}) \tag{2}$$

where $\bar{\mathbf{u}}$ and $\dot{\bar{\mathbf{u}}}$ are the "continuous" displacements and the "continuous" displacement rates; $[[\mathbf{u}]]$ and $[[\dot{\mathbf{u}}]]$ are the displacement jump and the displacement jump rate; H_S is a jump function, such that: $H_S(\mathbf{x}) = 0$ $\forall \mathbf{x} \in \Omega_h^- \cup \Omega^-$ and $H_S(\mathbf{x}) = 1$ $\forall \mathbf{x} \in \Omega_h^+ \cup \Omega^+$; $\hat{t}$ is the time and the dot above the variable means the time derivative (e.g. $\dot{\mathbf{u}} = \partial_{\hat{t}}\mathbf{u}$).

The strains are defined as the symmetric gradient (∇^s) of the displacement field. In the discrete approximation the strains are not defined at S, so the strain tensor ϵ is only defined in $\Omega \setminus S$ as

$$\epsilon(\mathbf{x}, \hat{t}) = \nabla^s \mathbf{u} = \nabla^s \bar{\mathbf{u}} + H_S \nabla^s [[\mathbf{u}]] = \bar{\epsilon} \quad \mathbf{x} \in \Omega \setminus S \tag{3}$$

where $\bar{\epsilon}$ is the "continuous" strain tensor, which is the strain occurring in the continuous part of the body. The corresponding strain rate tensor is defined as: $\dot{\epsilon}(\mathbf{x}, \hat{t}) = \nabla^s \dot{\mathbf{u}} = \nabla^s \dot{\bar{\mathbf{u}}} + H_S \nabla^s [[\dot{\mathbf{u}}]] = \dot{\bar{\epsilon}}$. The main characteristic of the discrete approximation is reminded by Eq. (3): the body behavior is established by means of strain–stress relationships outside the discontinuity $(\Omega \setminus S)$ and by displacement jump-traction relationships at the discontinuity (S).

2.2. Weak discontinuities

Consider a solid and homogenous body (Fig. 2) whose material points are labeled by the global coordinate system $\mathbf{x}$ (x, y). The body has a domain Ω and a boundary Γ $(\Gamma = \partial\Omega)$. This body has a discontinuity (strain localization zone) at S, which is contained in the subdomain Ω_h; the vector $\mathbf{n}$, normal to the discontinuity, defines a local coordinate system $\hat{\mathbf{n}}$ (n, t). The domain is divided into three subdomains: $\Omega = \Omega^- \cup \Omega_h \cup \Omega^+$; the subdomain Ω^+ is that located in the direction of $\mathbf{n}$. The lines S_h^- and S_h^+ bound the

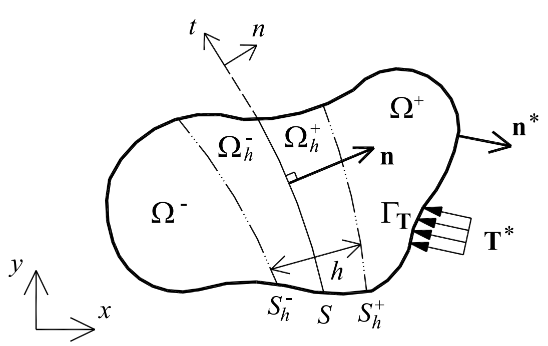

Fig. 1. Body with a strong discontinuity.

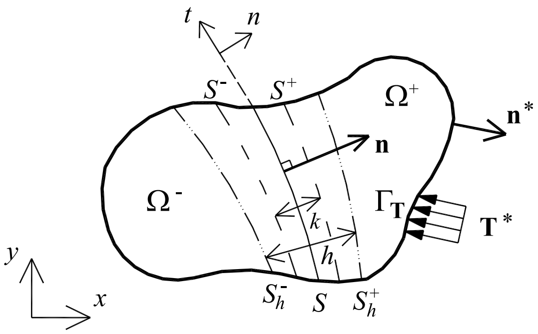

Fig. 2. Body with a weak discontinuity.

subdomain Ω_h and are separated by a distance h. The lines $S^-(n = n^-)$ and S^+ $(n = n^+)$ bound the subdomain Ω_k, which corresponds to the strain localization zone, and are separated by a distance k (localization zone width: $k = n^+ - n^-$); the subdomains Ω_k and Ω_h have the following relationships (Fig. 2): $\Omega_k \subset \Omega_h$ and $k < h$.

The displacement field $\mathbf{u}(\mathbf{x}, \hat{t})$ and the velocity $\dot{\mathbf{u}}(\mathbf{x}, \hat{t})$ are defined as

$$\mathbf{u}(\mathbf{x}, \hat{t}) = \bar{\mathbf{u}}(\mathbf{x}, \hat{t}) + H_k(\hat{\mathbf{n}})[[\mathbf{u}]](\mathbf{x}, \hat{t}) \tag{4}$$

$$\dot{\mathbf{u}}(\mathbf{x}, \hat{t}) = \frac{\partial \mathbf{u}(\mathbf{x}, \hat{t})}{\partial \hat{t}} = \dot{\bar{\mathbf{u}}}(\mathbf{x}, \hat{t}) + H_k(\hat{\mathbf{n}})[[\dot{\mathbf{u}}]](\mathbf{x}, \hat{t}) \tag{5}$$

where $\bar{\mathbf{u}}$ and $\dot{\bar{\mathbf{u}}}$ are the "continuous" displacements and the "continuous" displacement rates; $[[\mathbf{u}]]$ and $[[\dot{\mathbf{u}}]]$ are the displacement jump and the displacement jump rate; H_k is a ramp function defined as

$$H_k(\hat{\mathbf{n}}) = 0 \quad \forall n < n^-, \quad H_k(\hat{\mathbf{n}}) = \frac{n - n^-}{n^+ - n^-} \quad \forall n^- \leqslant n \leqslant n^+, \quad H_k(\hat{\mathbf{n}}) = 1 \quad \forall n > n^+$$

As before, the strains are the symmetric gradient (∇^s) of the displacement field. In the continuum approximation the strains are defined in the whole domain Ω and can be calculated as

$$\epsilon(\mathbf{x}, \hat{t}) = \nabla^s \mathbf{u} = \underbrace{\nabla^s \bar{\mathbf{u}} + H_k \nabla^s [[\mathbf{u}]]}_{\bar{\epsilon}} + \underbrace{\mu_k \frac{1}{k}([[\mathbf{u}]] \otimes \mathbf{n})^s}_{[[\epsilon]]} \tag{6}$$

where μ_k is a function defined as: $\mu_k = 1$ $\forall \mathbf{x} \in \Omega_k$ and $\mu_k = 0$ $\forall \mathbf{x} \in \Omega \setminus \Omega_k$; $\bar{\epsilon}$ is the "continuous" strain tensor, which is the strain occurring in the continuous part of the body; $[[\epsilon]]$ is the strain jump which takes place at the borders of the localization zone (S^+ and S^-) and is obtained by considering that: $\nabla H_k = \partial_{\mathbf{x}} H_k = \partial_{\hat{\mathbf{n}}} H_k \frac{\partial \mathbf{x}}{\partial \hat{\mathbf{n}}} = \frac{\mu_k}{k}\mathbf{n}$. The corresponding strain rate tensor is defined as: $\dot{\epsilon}(\mathbf{x}, \hat{t}) = \nabla^s \dot{\mathbf{u}} = \nabla^s \dot{\bar{\mathbf{u}}} + H_k \nabla^s [[\dot{\mathbf{u}}]] + \mu_k \frac{1}{k}([[\dot{\mathbf{u}}]] \otimes \mathbf{n})^s = \dot{\bar{\epsilon}} + [[\dot{\epsilon}]]$.

Remark 1. Outside the localization zone the "continuous" strains of both approximations are equal: $\bar{\epsilon}^d = \bar{\epsilon}^{wd}$ $\mathbf{x} \in \Omega \setminus \Omega_k$, but inside they are not: $\bar{\epsilon}^d \neq \bar{\epsilon}^{wd}$ $\mathbf{x} \in \Omega_k$; however, if the softening band zone width is very small $(k \to 0)$, then $\bar{\epsilon}^d \approx \bar{\epsilon}^{wd}$ $\mathbf{x} \in \Omega_k$ because $H_S \approx H_k$. The superscript "d" refers to the discrete discontinuity approximation and "wd" refers to the weak discontinuity approximation.

It is important to point out the characteristics of the strain jump that emerge from its definition (6). For this purpose, let express the strain jump in the local three dimensional coordinate system

$$[[\boldsymbol{\epsilon}]] = \begin{bmatrix} [[\epsilon]]_{nn} & [[\epsilon]]_{nt} & [[\epsilon]]_{ns} \\ [[\epsilon]]_{tn} & [[\epsilon]]_{tt} & [[\epsilon]]_{ts} \\ [[\epsilon]]_{sn} & [[\epsilon]]_{st} & [[\epsilon]]_{ss} \end{bmatrix} = \frac{1}{k} \begin{bmatrix} [[u]]_n & \frac{1}{2}[[u]]_t & \frac{1}{2}[[u]]_s \\ \frac{1}{2}[[u]]_t & 0 & 0 \\ \frac{1}{2}[[u]]_s & 0 & 0 \end{bmatrix} \tag{7}$$

From (7) may be observed that the kinematics of bodies with weak discontinuities establishes the following conditions for the strain jump terms:

(1) $[[\epsilon]]_{tt}$, $[[\epsilon]]_{ts}$ and $[[\epsilon]]_{ss}$ must be zero: $[[\epsilon]]_{tt} = [[\epsilon]]_{ts} = [[\epsilon]]_{ss} = 0$; this is called by Oliver (1996) "Strong Discontinuity Condition". In this paper this will be referred as kinematics condition of discontinuities.
(2) $[[\epsilon]]_{nn}$, $[[\epsilon]]_{nt}$ and $[[\epsilon]]_{ns}$ can be different from zero. The component $[[u]]_n$ of the displacement jump, which corresponds to a Mode I in fracture mechanics, only contributes to $[[\epsilon]]_{nn}$. The component $[[u]]_t$ of the displacement jump, which corresponds to a Mode II in fracture mechanics, only contributes to $[[\epsilon]]_{nt}$ and $[[\epsilon]]_{tn}$. The component $[[u]]_s$ of the displacement jump, which corresponds to a Mode III in fracture mechanics, only contributes to $[[\epsilon]]_{ns}$ and $[[\epsilon]]_{sn}$.

Both conditions are important, however here only will be dealt with the first because the need of its fulfillment is the motivation to propose in this work an anisotropic damage model for the continuum approximation. The second condition is also very important and will be used to identify some problems with the finite element approximation; this will be presented in a forthcoming paper.

3. Constitutive model

In this paper, the constitutive behavior of the discontinuity is modeled by using a family of damage constitutive models and considering that the material behavior is rate-independent and local and follows the infinitesimal strain theory. Continuum damage mechanics is based on the thermodynamics of irreversible processes and the theory of the internal state variable (Simo and Ju, 1987; Mazars and Pijaudier-Cabot, 1989). Basically, there are two types of damage models: (1) isotropic damage, which uses a scalar damage variable and (2) anisotropic damage, which can use a damage tensor. In this paper, an isotropic damage model is developed to model discontinuities by the discrete approximation and an anisotropic damage model by the continuum approximation.

In this section, the basic concepts of a classic isotropic damage model are summarized as follows:

- Effective stress

$$\boldsymbol{\sigma} = (1-d)\bar{\boldsymbol{\sigma}} = \frac{\partial \Psi(\boldsymbol{\epsilon}, r)}{\partial \boldsymbol{\epsilon}} = (1-d)\mathbf{C} : \boldsymbol{\epsilon} \tag{8}$$

- Free energy function

$$\Psi(\boldsymbol{\epsilon}, r) = [1 - d(r)]\Psi^{\mathrm{e}}(\boldsymbol{\epsilon}), \quad \Psi^{\mathrm{e}}(\boldsymbol{\epsilon}) = \frac{1}{2}\boldsymbol{\epsilon} : \mathbf{C} : \boldsymbol{\epsilon} \tag{9}$$

- Damage criterion

$$f(\boldsymbol{\sigma}, q) = \tau_{\boldsymbol{\sigma}} - q \quad g(\boldsymbol{\epsilon}, r) = \tau_{\boldsymbol{\epsilon}} - r \tag{10}$$

- Damage variable

$$d = 1 - \frac{q(r)}{r}\frac{1}{E} \tag{11}$$

- Evolution law

$$\dot{r} = \lambda \tag{12}$$

- Hardening law

$$\dot{q} = \mathscr{H}\dot{r} \tag{13}$$

- Kuhn–Tucker conditions and the consistency requirement

$$\lambda \geqslant 0, \quad f(\boldsymbol{\sigma}, q) \leqslant 0, \quad \lambda f(\boldsymbol{\sigma}, q) = 0 \tag{14}$$

$$\lambda \dot{f}(\boldsymbol{\sigma}, q) = 0 \tag{15}$$

In the above equations d is the damage variable ($d \in [0, 1]$) and is associated with an irreversible process, such that $\dot{d} \geqslant 0$; the stress tensor $\boldsymbol{\sigma}$ is a function of the strain tensor $\boldsymbol{\epsilon}$ and can be calculated by means of the free energy function (per unit mass) $\Psi(\epsilon, r)$ (9); Ψ^{e} is the elastic free energy; $\mathbf{C}$ is the elastic constitutive tensor, defined as: $\mathbf{C} = \hat{\lambda}\mathbf{1} \otimes \mathbf{1} + 2\hat{\mu}\mathbf{I}$, $\hat{\lambda}$ and $\hat{\mu}$ are Lame's constants, $\mathbf{1}$ is the second order identity tensor ($\mathbf{1} = \delta_{ij}$; δ_{ij} is the Kronecker Delta) and $\mathbf{I}$ is the fourth order identity tensor ($\mathbf{I} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$); the function f defines the failure criterion in the stress space ($f(\boldsymbol{\sigma}, q) \leqslant 0$); $\tau_{\boldsymbol{\sigma}}$ is a norm and q is a stress type internal variable; the function g establishes the failure criterion in the strain space $(g(\epsilon, r) \leqslant 0)$. τ_{ϵ} is a norm and r is a strain type internal variable ($r = \max\{r_0, \max(\tau_{\epsilon})\}$; where r_0 is the initial value of r, such that $r \in [r_0, \infty)$); E is the modulus of elasticity; λ is a parameter called damage multiplier; $\mathscr{H}$ is the hardening/softening modulus.

4. Isotropic damage models

4.1. Discrete approximation

The constitutive model for the discrete approximation uses the displacement jump as independent variable. This model is rigid-perfect at the beginning of the response when the discontinuity is not formed, thus some terms in the constitutive tensor are not bounded. It is inconvenient to regard this constitutive tensor as the initial constitutive tensor and to try to use a damage variable "d" with a range $d \in [0, 1]$; therefore, it is defined an "initial" constitutive tensor $\mathbf{C}_S^0$ associated to an intermediate state between the no damaged and the completely damaged, which is proposed as

$$\mathbf{C}_S^0 = \mathbf{n} \otimes \mathbf{n}E + \mathbf{t} \otimes \mathbf{t}G + \mathbf{s} \otimes \mathbf{s}G \tag{16}$$

where $\mathbf{n}$, $\mathbf{t}$, $\mathbf{s}$ are unit vectors in the direction of the local axes n, t, s. The "initial" constitutive tensor is defined in a convenient way, such that it is easy to establish a relationship between this constitutive model and the anisotropic damage model for the continuum approximation. The proposed "initial" constitutive tensor is similar to that obtained by Oliver (2000) for the discrete model as a projection of the strong discontinuity approximation model: $\mathbf{C}_S^0 = \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} = (\frac{E(1-\nu)}{(1+\nu)(1-2\nu)})\mathbf{n} \otimes \mathbf{n} + G\mathbf{t} \otimes \mathbf{t} + G\mathbf{s} \otimes \mathbf{s}$; if the Poisson's ratio is zero ($\nu = 0$) both "initial" constitutive tensors are equal.

The constitutive tensor $\mathbf{C}_S$ is defined as

$$\mathbf{C}_S = (1 - d^{\mathrm{d}})\mathbf{C}_S^0 \tag{17}$$

where d^{d} is a damage variable (scalar) for the discrete approximation, to be defined later.

The free energy (per unit mass) Ψ^{d} is defined as

$$\Psi^{\mathrm{d}} = (1 - d^{\mathrm{d}})\tfrac{1}{2}[[\mathbf{u}]] \cdot \mathbf{C}_S^0 \cdot [[\mathbf{u}]] \tag{18}$$

The traction vector can be calculated from the free energy as

$$\mathbf{T} = \frac{\partial \Psi^{\mathrm{d}}}{\partial [[\mathbf{u}]]} = (1 - d^{\mathrm{d}})\mathbf{C}_S^0 \cdot [[\mathbf{u}]] \tag{19}$$

The damage function f defines a failure criterion in the traction space and is given by

$$f(\tau_{\mathbf{T}}, q^{\mathrm{d}}) = \tau_{\mathbf{T}} - q^{\mathrm{d}} \leqslant 0 \tag{20}$$

where $\tau_{\mathbf{T}}$ is the norm of the tractions and is defined as

$$\tau_{\mathbf{T}} = \sqrt{\mathbf{T} \cdot \mathbf{W}_{[[\mathbf{u}]]} \cdot \mathbf{T}} \tag{21}$$

q^{d} is a traction type internal variable and $\mathbf{W}_{[[\mathbf{u}]]}$ is a second order weight tensor.

The traction is associated with the plane that corresponds to the discontinuity. However, the orientation of the discontinuity is unknown and it cannot be calculated using the damage function f. So it is necessary to establish a complementary criterion that allows the determination of the orientation of the discontinuity; e.g. the maximum principal stress or the maximum shear stress.

As above, it is also possible to define another damage function, g, to establish a damage criterion in the displacement jump space

$$g(\tau_{[[\mathbf{u}]]}, r^{\mathrm{d}}) = \tau_{[[\mathbf{u}]]} - r^{\mathrm{d}} \leqslant 0 \tag{22}$$

where $\tau_{[[\mathbf{u}]]}$ is the norm of the displacement jumps and is defined as

$$\tau_{[[\mathbf{u}]]} = \sqrt{[[\mathbf{u}]] \cdot \mathbf{W}_{[[\mathbf{u}]]} \cdot [[\mathbf{u}]]} \tag{23}$$

r^{d} is a displacement jump type internal variable and $\mathbf{W}_{[[\mathbf{u}]]}$ is a second order weight tensor defined as

$$\mathbf{W}_{[[\mathbf{u}]]} = \begin{bmatrix} W_{nn} & 0 & 0 \\ 0 & W_{tt} & 0 \\ 0 & 0 & W_{ss} \end{bmatrix} \tag{24}$$

with

$$\begin{aligned} W_{nn} &= \begin{cases} 1 & \text{if } T_n \geqslant 0 \text{ and } [[u]]_n \geqslant 0 \\ 0 & \text{if } T_n < 0 \end{cases} \\ W_{tt} &= \frac{(([[u]]_n)_{\mathrm{crit}})^2}{(([[u]]_t)_{\mathrm{crit}})^2} \\ W_{ss} &= \frac{(([[u]]_n)_{\mathrm{crit}})^2}{(([[u]]_s)_{\mathrm{crit}})^2} \end{aligned} \tag{25}$$

where $([[u]]_n)_{\mathrm{crit}}$, $([[u]]_t)_{\mathrm{crit}}$ and $([[u]]_s)_{\mathrm{crit}}$ are the critical values of the displacement jump for Modes I, II, and III, respectively, and correspond to the displacement jump values when the discontinuity is unable to transfer tractions. Also, the quantities W_{nn}, W_{tt} and W_{ss} can be simply considered as values to weigh the mode of failure in the failure criterion; for example, if it is considered that the failure is dominated just by the Mode I of fracture, then $W_{nn} = 1$ and $W_{tt} = W_{ss} = 0$. Fig. 3 illustrates the value of $([[u]]_n)_{\mathrm{crit}}$ and the loading and unloading trajectories for the Mode I of fracture.

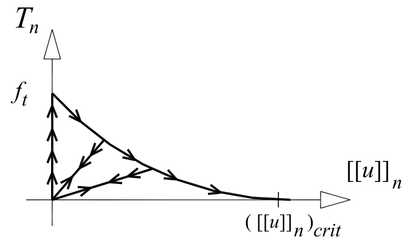

Fig. 3. Definition of the softening curve: $T_n - [[u]]_n$.

The displacement jump type internal variable $r^{\rm d}$ is defined as

$$r^{\rm d} = \max\left\{ r_0^{\rm d}, \underbrace{\max(\tau_{[[\mathbf{u}]]})}_{s\in[0,\hat{t}]} \right\} \tag{26}$$

where $r_0^{\rm d}$ is the initial value of $r^{\rm d}$, here taking the value $r_0^{\rm d} = 0$, thus the $r^{\rm d}$ range is $r^{\rm d} \in [0, +\infty)$.

The damage variable $d^{\rm d}$ is related to the internal variables $r^{\rm d}$ and $q^{\rm d}$ by Eq. (11): $d = 1 - \frac{q^{\rm d}(r^{\rm d})}{r^{\rm d}}\frac{1}{E}$. Considering that the ranges of the internal variables are $r^{\rm d} \in [0, +\infty)$ and $q^{\rm d} \in [0, f_t]$, where f_t is the resistant strength, the range of the damage variable for this model is: $d^{\rm d} \in (-\infty, 1]$. It can be observed that damage variable range is not as the traditional continuum type models ($d \in [0, 1]$).

The damage function g that defines the failure surface is illustrated in Fig. 4. The failure surface initiates as a point at the origin ($|[[\mathbf{u}]]| = 0$ and $r^{\rm d} = 0$). Then the surface grows in proportion to $([[u]])_{\rm crit}$ when $r^{\rm d}$ is incremented.

The tangent constitutive operator is obtained from the constitutive relationship (19) by deriving it with respect to the time

$$\dot{\mathbf{T}} = (1 - d^{\rm d})\mathbf{C}_S^0 \cdot [[\dot{\mathbf{u}}]] - \dot{d}^{\rm d}\mathbf{C}_S^0 \cdot [[\mathbf{u}]] \tag{27}$$

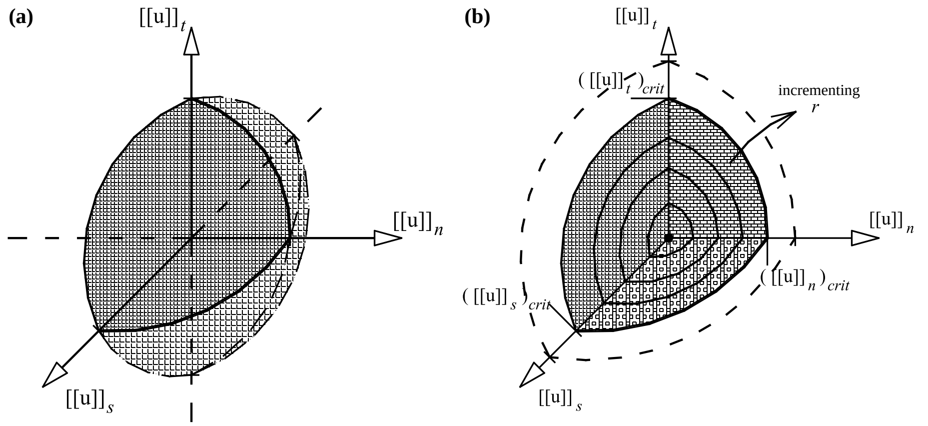

Fig. 4. Damage criterion for the discrete approximation: (a) failure surface and (b) damage evolution.

The term $\dot{d}^{\mathrm{d}}$ is calculated as

$$\begin{aligned} \dot{d}^{\mathrm{d}} &= \frac{\partial d^{\mathrm{d}}}{\partial r^{\mathrm{d}}}\dot{r}^{\mathrm{d}} = \left((1-d^{\mathrm{d}}) - \frac{\mathscr{H}^{\mathrm{d}}}{E}\right)\dot{r}^{\mathrm{d}} \\ \dot{r}^{\mathrm{d}} &= \dot{\tau}_{[[\mathbf{u}]]} = \frac{1}{\tau_{[[\mathbf{u}]]}}[[\mathbf{u}]]\cdot\mathbf{W}_{[[\mathbf{u}]]}\cdot[[\dot{\mathbf{u}}]] \end{aligned} \tag{28}$$

The tangent constitutive operators, which relates the traction rate vector $\dot{\mathbf{T}}$ with the displacement jump rate vector $[[\dot{\mathbf{u}}]]$, are obtained by substituting (28_1) and (28_2) in (27):

- For the elastic loading and unloading range ($\dot{d}^{\mathrm{d}} = 0$)

$$\dot{\mathbf{T}} = (1-d^{\mathrm{d}})\mathbf{C}_S^0\cdot[[\dot{\mathbf{u}}]] \tag{29}$$

- For the nonlinear and neutral loading range ($\dot{d}^{\mathrm{d}} > 0$)

$$\dot{\mathbf{T}} = \left[(1-d^{\mathrm{d}})\mathbf{C}_S^0 + \left(-(1-d^{\mathrm{d}}) + \frac{\mathscr{H}^{\mathrm{d}}}{E}\right)\left(\frac{1}{(\tau_{[[\mathbf{u}]]})^2}\right)(\mathbf{C}_S^0\cdot[[\mathbf{u}]])\otimes([[\mathbf{u}]]\cdot\mathbf{W}_{[[\mathbf{u}]]})\right]\cdot[[\dot{\mathbf{u}}]] \tag{30}$$

Remark 2. Eqs. (29) and (30) are only valid for $[[u]]_n \geqslant 0$. This is due to the physical sense of the Mode I of fracture; there are just two options: the crack is opened ($[[u]]_n > 0$) or is closed ($[[u]]_n = 0$). On the other hand, the values of $[[u]]_t$ and $[[u]]_s$ can be positive or negative depending on the sign of the traction components T_t and T_s, respectively.

If the traction component T_n is negative, it will be necessary to modify the value of the constitutive tensor term $(C_S^0)_{nn} = (1-d^{\mathrm{d}})E$ by an unbounded value: $(C_S^0)_{nn} = \infty$. Then the same procedure and equations established in this section may be used in this model.

Remark 3. If $T_n < 0$ the constitutive model is no longer isotropic, becoming anisotropic. However, in this paper this model is referred as isotropic damage model by considering the case $T_n > 0$.

Summarizing. In this isotropic damage model for discrete approximation of discontinuities, the independent variables are the strain ϵ ($\epsilon = \bar{\epsilon}$) at $\Omega \setminus S$ and the displacement jump $[[\mathbf{u}]]$ at S. The dependent variables are the stress $\boldsymbol{\sigma}$ at $\Omega \setminus S$ and the traction $\mathbf{T}$ at S. In this model the damage is localized at S and there is no damage at $\Omega \setminus S$. f_t and $\mathscr{H}^{\mathrm{d}}$ are considered material properties.

4.2. Continuum approximation

This subsection analyzes whether the isotropic damage model is suitable to reproduce the constitutive behavior of weak discontinuities. For this purpose, consider the constitutive model described in Section 2: Eqs. (8)–(15). Fig. 5 illustrates the behavior of the model for the one dimensional case. It may be observed that the initial response is not perfectly rigid as in the discrete approximation (Fig. 3). To evaluate the model it should be noticed that the description given in Section 2 was not complete, it is still needed the definition of the norms; in this paper, the following definitions are used: $\tau_{\sigma} = \sqrt{\boldsymbol{\sigma}:\boldsymbol{\sigma}}$ and $\tau_{\epsilon} = \sqrt{\epsilon:\epsilon}$.

This analysis starts from the stress–strain relationship, which is established from (6) and (8)

$$\left(1 + \frac{d^{\mathrm{wd}}}{1-d^{\mathrm{wd}}}\right)\boldsymbol{\sigma} = \mathbf{C}:\left[\bar{\epsilon} + \frac{1}{k}([[\mathbf{u}]]\otimes\mathbf{n})^{\mathrm{s}}\right] \tag{31}$$

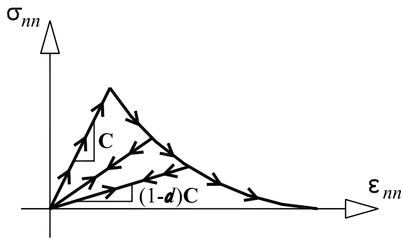

Fig. 5. Stress–strain curve of the material at the localization zone using a weak discontinuity approximation.

This equation is fulfilled if the following two equations are valid:

$$\boldsymbol{\sigma} = \mathbf{C} : \bar{\boldsymbol{\epsilon}} \tag{32}$$

$$\left(\frac{d^{\mathrm{wd}}}{1-d^{\mathrm{wd}}}\right)\boldsymbol{\sigma} = \mathbf{C} : \left[\frac{1}{k}([[\mathbf{u}]] \otimes \mathbf{n})^{\mathrm{s}}\right] \Rightarrow \left(\frac{d^{\mathrm{wd}}}{1-d^{\mathrm{wd}}}\right)\bar{\boldsymbol{\epsilon}} = \frac{1}{k}([[\mathbf{u}]] \otimes \mathbf{n})^{\mathrm{s}} \tag{33}$$

Eq. (32) establishes the elastic (undamaged) standard relationship between the "continuous" strains $\bar{\boldsymbol{\epsilon}}$ and the stresses $\boldsymbol{\sigma}$. This equation implies that outside the strain localization zone the material behaves elastic and there is no damage. Therefore it may be assumed that the nonlinear behavior, related to the damage, is produced in the localization zone and is associated to the strain jump $[[\boldsymbol{\epsilon}]]$ (33_1). Eq. (33_2), which is obtained by substituting (32) in (33_1), establishes the following condition:

$$\bar{\epsilon}_{tt} = \bar{\epsilon}_{ts} = \bar{\epsilon}_{ss} = 0 \tag{34}$$

This condition, resulting from the kinematics condition, can be considered a limitation for the implementation of this constitutive model because the "continuous" strain tensor $\bar{\boldsymbol{\epsilon}}$ does not always fulfills it. To overcome this problem, in this paper an anisotropic damage model that always fulfill this condition is developed.

5. Anisotropic damage model

The application of isotropic damage models to the continuum approximation of discontinuities has the drawback of not always fulfilling the kinematics condition when failure occurs. For this reason, in this section an anisotropic damage model is developed, which should be suitable to model discontinuities. In this paper, a constitutive model is considered to be suitable when it is capable of fulfilling the following two conditions:

(1) *Null tractions*. If the material is completely damaged, i.e. $d^{\mathrm{wd}} = 1$, then the discontinuity should not transfer tractions; therefore: $\sigma_{nn} = \sigma_{nt} = \sigma_{ns} = 0$.
(2) *Kinematics condition of discontinuities* (strong discontinuity conditions): $[[\epsilon]]_{tt} = [[\epsilon]]_{ss} = [[\epsilon]]_{ts} = 0$.

Firstly, before developing an anisotropic damage model for weak discontinuities, two anisotropic damage models quoted in Simo and Ju (1987) were evaluated to determine whether they are suitable to model discontinuities. These models come from the effective stresses concept ($\boldsymbol{\sigma} = \mathbf{M} : \bar{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma} = \mathbf{M} : \mathbf{C} : \boldsymbol{\epsilon}$) and effective strains concept ($\breve{\boldsymbol{\epsilon}} = \mathbf{M} : \boldsymbol{\epsilon}$ and $\boldsymbol{\sigma} = \mathbf{C} : \mathbf{M} : \boldsymbol{\epsilon}$); where $\mathbf{M}$ is a fourth order tensor that characterizes the damage, $\mathbf{C}$ is the elastic constitutive tensor, $\bar{\boldsymbol{\sigma}}$ is effective stress and $\breve{\boldsymbol{\epsilon}}$ is the effective strain. In this paper, the following definition of the damage tensor is proposed:

$$\mathbf{M} = \mathbf{I} - d^{\text{wd}}(\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{t}\otimes\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n}\otimes\mathbf{t}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{s}\otimes\mathbf{n}\otimes\mathbf{s} + \mathbf{s}\otimes\mathbf{n}\otimes\mathbf{s}\otimes\mathbf{n}) \tag{35}$$

where d^{wd} is the damage variable (scalar) for the continuum approximation with weak discontinuities. Unfortunately, in general this two models does not fulfill these two conditions simultaneously. The effective stress model fulfills the null traction condition ($d^{\text{wd}} = 1 \Rightarrow \sigma_{nn} = \sigma_{nt} = \sigma_{ns} = 0$), but does not fulfill the kinematics condition because in general $[[\epsilon]]_{tt} \neq 0$ and $[[\epsilon]]_{ss} \neq 0$. The effective strain model fulfills the kinematic condition ($[[\epsilon]]_{tt} = [[\epsilon]]_{ss} = [[\epsilon]]_{ts} = 0$), but does not fulfill the null traction condition (if $\epsilon_{tt} \neq 0$ and/or $\epsilon_{ss} \neq 0$, it may be obtained that $\sigma_{nn} \neq 0$ for $d^{\text{wd}} = 1$).

After finding out that these models are not suitable to model discontinuities, a third model that fulfills both conditions simultaneously is presented. To simplify the presentation of the model, the constitutive equation is written [2] term by term in a local coordinate system and using the form $\boldsymbol{\epsilon} = [\mathbf{C}_S]^{-1} : \boldsymbol{\sigma}$

$$\begin{aligned}
&\blacktriangleright \quad \epsilon_{nn} = \frac{1}{(1-d^{\text{wd}})E}\sigma_{nn} - \frac{v}{E}\sigma_{tt} - \frac{v}{E}\sigma_{ss}\\
&\qquad \epsilon_{tt} = -\frac{v}{E}\sigma_{nn} + \frac{1}{E}\sigma_{tt} - \frac{v}{E}\sigma_{ss}\\
&\qquad \epsilon_{ss} = -\frac{v}{E}\sigma_{nn} - \frac{v}{E}\sigma_{tt} + \frac{1}{E}\sigma_{ss}\\
&\blacktriangleright \quad \epsilon_{nt} = \frac{1}{(1-d^{\text{wd}})(2G)}\sigma_{nt}\\
&\blacktriangleright \quad \epsilon_{ns} = \frac{1}{(1-d^{\text{wd}})(2G)}\sigma_{ns}\\
&\qquad \epsilon_{ts} = \frac{1}{2G}\sigma_{ts}
\end{aligned} \tag{36}$$

In this model the strain jump $[[\boldsymbol{\epsilon}]]$ is associated with the "damaged" part of the strains ($d^{\text{wd}}\boldsymbol{\epsilon}$). To identify and analyze the "undamaged" and the "damaged" part of the strain term ϵ_{nn}, Eq. (36_1) is divided in two parts

$$\epsilon_{nn} = \epsilon_{nn}^a + \epsilon_{nn}^b \tag{37}$$

where

$$\epsilon_{nn}^a = \frac{1}{(1-d^{\text{wd}})E}\sigma_{nn} \quad \text{and} \quad \epsilon_{nn}^b = -\frac{v}{E}\sigma_{tt} - \frac{v}{E}\sigma_{ss} \tag{38}$$

From Eqs. (37) and (38) may be identified the damaged part of the strain term ϵ_{nn}: $[[\epsilon]]_{nn}$ (associated to the Mode I of failure), and the undamaged part: $\bar{\epsilon}_{nn}$, such that

$$\epsilon_{nn} = \bar{\epsilon}_{nn} + [[\epsilon]]_{nn} \quad \begin{cases} \bar{\epsilon}_{nn} = (1-d^{\text{wd}})\epsilon_{nn}^a + \epsilon_{nn}^b \\ [[\epsilon]]_{nn} = d^{\text{wd}}\epsilon_{nn}^a \end{cases} \tag{39}$$

The identification of the "damaged" (associated to the Mode II: $[[\epsilon]]_{nt}$ and the Mode III: $[[\epsilon]]_{ns}$ of failure) and the "undamaged" ($\bar{\epsilon}_{nt}$ and $\bar{\epsilon}_{ns}$) parts of the strain terms ϵ_{nt} and ϵ_{ns} is straightforward

$$\epsilon_{nt} = \bar{\epsilon}_{nt} + [[\epsilon]]_{nt} \quad \begin{cases} \bar{\epsilon}_{nt} = (1-d^{\text{wd}})\epsilon_{nt} \\ [[\epsilon]]_{nt} = d^{\text{wd}}\epsilon_{nt} \end{cases} \tag{40}$$

[2] The black triangles point out which terms are function of the damage variable d^{wd}.

$$\epsilon_{ns} = \bar{\epsilon}_{ns} + [[\epsilon]]_{ns} \quad \begin{cases} \bar{\epsilon}_{ns} = (1 - d^{\mathrm{wd}})\epsilon_{ns} \\ [[\epsilon]]_{ns} = d^{\mathrm{wd}}\epsilon_{ns} \end{cases} \tag{41}$$

Remark 4. From the absence of the damage variable in equations (36_2), (36_3) and (36_6), it may be demonstrated that the following terms of the strain jump tensor are zero: $[[\epsilon]]_{tt} = [[\epsilon]]_{ss} = [[\epsilon]]_{ts} = 0$. Therefore, the kinematics condition of discontinuities is fulfilled.

Remark 5. Given that the terms of the "continuous" strain tensor $\bar{\epsilon}$ are bounded, it may be demonstrated that this model fulfill the null traction condition as follows:

From (36_1) and (39) : $\sigma_{nn} = (1 - d^{\mathrm{wd}})(E)\epsilon^a_{nn}$, if $d^{\mathrm{wd}} = 1$ then $\sigma_{nn} = 0$

From (36_4) and (40) : $\sigma_{nt} = (1 - d^{\mathrm{wd}})(2G)\epsilon_{nt}$, if $d^{\mathrm{wd}} = 1$ then $\sigma_{nt} = 0$

From (36_5) and (41) : $\sigma_{ns} = (1 - d^{\mathrm{wd}})(2G)\epsilon_{ns}$, if $d^{\mathrm{wd}} = 1$ then $\sigma_{ns} = 0$

Remark 6. If the terms ϵ_{nt} and ϵ_{ns} were bounded and $d^{\mathrm{wd}} = 1$, then $\bar{\epsilon}_{nt} = \bar{\epsilon}_{ns} = 0$ (Eqs. (40) and (41)), i.e. the shear components of the "continuous" strain tensor related to the strain jump are always zero. But if the term ϵ_{nn} was bounded and $d^{\mathrm{wd}} = 1$, then, in general, $\bar{\epsilon}_{nn} \neq 0$.

Now that the suitability of the proposed anisotropic damage model for the constitutive modeling of weak discontinuities has been demonstrated, in what follows the characteristics of this model will be presented. The free energy (per unit mass) is defined as: $\Psi^{\mathrm{wd}} = \frac{1}{2}\boldsymbol{\epsilon} : \mathbf{C}_S : \boldsymbol{\epsilon}$, where $\mathbf{C}_S$ is the fourth order constitutive tensor, defined by (36). In Appendix A the constitutive matrix for the plane stress and plane strain cases are presented. From Appendix A it may be observed that the constitutive matrices in this model are symmetric. The stress tensor is calculated from (8): $\boldsymbol{\sigma} = \partial_{\epsilon}\Psi^{\mathrm{wd}}(\boldsymbol{\epsilon}, r) = \mathbf{C}_S : \boldsymbol{\epsilon}$. The failure function f, defined as (10): $f(\boldsymbol{\sigma}, q^{\mathrm{wd}}) = \tau_{\boldsymbol{\sigma}} - q^{\mathrm{wd}}$, is a function of the norm $\tau_{\boldsymbol{\sigma}}$ and the stress type internal variable q^{wd}. For this model the norm is defined as

$$\tau_{\boldsymbol{\sigma}} = \sqrt{\boldsymbol{\sigma}^{\circledast} : \mathbf{W}_{\epsilon} : \boldsymbol{\sigma}^{\circledast}} \tag{42}$$

where $\mathbf{W}_{\epsilon}$ is a fourth order tensor and $\boldsymbol{\sigma}^{\circledast}$ is a second order tensor, whose terms correspond to the components σ_{nn}, σ_{nt}, σ_{ns}, defined as: $\boldsymbol{\sigma}^{\circledast} = \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} : \sigma + \mathbf{H}^{\circledast} : \boldsymbol{\sigma}$, $\mathbf{H}^{\circledast}$ is a fourth order tensor to be defined later in this paper.

The damage function g, defined as (10): $g(\tau_{\epsilon}, r^{\mathrm{wd}}) = \tau_{\epsilon} - r^{\mathrm{wd}} \leqslant 0$, is a function of the norm τ_{ϵ} and the strain type internal variable r^{wd}. For this model the norm is defined as

$$\tau_{\epsilon} = \sqrt{\boldsymbol{\epsilon}^{\circledast} : \mathbf{W}_{\epsilon} : \boldsymbol{\epsilon}^{\circledast}} \tag{43}$$

where $\boldsymbol{\epsilon}^{\circledast}$ is a second order tensor with components corresponding to those of the strain tensor associated to the damage and in consequence to the non null components of the strain jump (see Eqs. (39)–(41))

$$\boldsymbol{\epsilon}^{\circledast} = \begin{bmatrix} \epsilon^a_{nn} & \epsilon_{nt} & \epsilon_{ns} \\ \epsilon_{tn} & 0 & 0 \\ \epsilon_{sn} & 0 & 0 \end{bmatrix} = \frac{1}{d^{\mathrm{wd}}}[[\boldsymbol{\epsilon}]] \tag{44}$$

Considering the definition of $\boldsymbol{\epsilon}^{\circledast}$ (44), the norm τ_{ϵ} (43) can be written as

$$\tau_{\epsilon} = \frac{1}{d^{\mathrm{wd}}}\sqrt{[[\boldsymbol{\epsilon}]] : \mathbf{W}_{\epsilon} : [[\boldsymbol{\epsilon}]]} \tag{45}$$

where $\mathbf{W}_\epsilon$ is a fourth order tensor defined as

$$\mathbf{W}_\epsilon = W_{nnnn}\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} + W_{ntnt}\mathbf{n}\otimes\mathbf{t}\otimes\mathbf{n}\otimes\mathbf{t} + W_{ntnt}\mathbf{t}\otimes\mathbf{n}\otimes\mathbf{t}\otimes\mathbf{n} + W_{nsns}\mathbf{n}\otimes\mathbf{s}\otimes\mathbf{n}\otimes\mathbf{s} + W_{nsns}\mathbf{s}\otimes\mathbf{n}\otimes\mathbf{s}\otimes\mathbf{n} \tag{46}$$

with

$$W_{nnnn} = \begin{cases} 1 & \text{if } \sigma_{nn} \geqslant 0 \text{ or } \epsilon^a_{nn} \geqslant 0 \\ 0 & \text{if } \sigma_{nn} < 0 \text{ or } \epsilon^a_{nn} \leqslant 0 \end{cases}$$
$$W_{ntnt} = \frac{((\epsilon^a_{nn})_{\text{crit}})^2}{2((\epsilon_{nt})_{\text{crit}})^2} \tag{47}$$
$$W_{nsns} = \frac{((\epsilon^a_{nn})_{\text{crit}})^2}{2((\epsilon_{ns})_{\text{crit}})^2}$$

The values of $(\epsilon^a_{nn})_{\text{crit}}$, $(\epsilon_{nt})_{\text{crit}}$ and $(\epsilon_{ns})_{\text{crit}}$ correspond to those of the strain terms ϵ^a_{nn}, ϵ_{nt} and ϵ_{ns} when the tractions are null ($q^{\text{wd}} = 0$) at the localization zone boundaries. These values depend on the localization zone width, k, and may be related to the displacement jump of the discrete approximation model. The quantities of W_{nnnn}, W_{ntnt} and W_{nsns}.can be simply considered as values to weigh the mode of failure in the failure criterion.

From the definition of $\boldsymbol{\epsilon}^{\circledast}$ (44) may be observed that the terms $\epsilon^{\circledast}_{nt}$ and $\epsilon^{\circledast}_{ns}$ correspond to the terms $\epsilon_{nt} y \epsilon_{ns}$ of the strain tensor $\boldsymbol{\epsilon}$, while the term $\epsilon^{\circledast}_{nn}$ corresponds to just a part of the strain term ϵ_{nn} (ϵ^a_{nn}). Unfortunately, ϵ^a_{nn} cannot be calculated only from the strain tensor; in this paper, two ways of calculating ϵ^a_{nn} are presented

$$\epsilon^a_{nn} = \frac{1}{(1-d^{\text{wd}})E}\mathbf{n}\otimes\mathbf{n} : \mathbf{C}_S : \boldsymbol{\epsilon}$$
$$\epsilon^a_{nn} = \epsilon_{nn} + \frac{\nu}{E}\mathbf{t}\otimes\mathbf{t} : \mathbf{C}_S : \boldsymbol{\epsilon} + \frac{\nu}{E}\mathbf{s}\otimes\mathbf{s} : \mathbf{C}_S : \boldsymbol{\epsilon} \tag{48}$$

If (48_1) is used [3], then the strain tensor $\boldsymbol{\epsilon}^{\circledast}$ can be calculated as

$$\boldsymbol{\epsilon}^{\circledast} = \frac{1}{(1-d^{\text{wd}})E}\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} : \mathbf{C}_S : \boldsymbol{\epsilon} + \mathbf{H}^{\circledast} : \boldsymbol{\epsilon} \tag{49}$$

with

$$\mathbf{H}^{\circledast} = \mathbf{n}\otimes\mathbf{t}\otimes\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n}\otimes\mathbf{t}\otimes\mathbf{n} + \mathbf{s}\otimes\mathbf{n}\otimes\mathbf{s}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{s}\otimes\mathbf{n}\otimes\mathbf{s} \tag{50}$$

Remark 7. Calculating ϵ^a_{nn} and consequently $\boldsymbol{\epsilon}^{\circledast}$ has the inconvenient that the value of the damage variable $d^{\text{wd}}(r^{\text{wd}})$ a priori should be known. However, r^{wd} is a function of τ_ϵ and consequently of ϵ^a_{nn}, therefore, in the numerical implementation of this model the use of a nonlinear iterative procedure to calculate τ_ϵ and later d^{wd} is needed.

The failure function g is a function of the strain type internal variable r^{wd} and is calculated as: $r^{\text{wd}} = \max\{r^{\text{wd}}_0, \max(\tau_\epsilon)\}$, where r^{wd}_0 is the initial value of r^{wd}; the range of r^{wd} is $r^{\text{wd}} \in [r^{\text{wd}}_0, \infty)$. Fig. 6 illustrates the failure surface and its evolution when r^{wd} increases. In this figure, the axes that define the space are ϵ^a_{nn}, ϵ_{nt} and ϵ_{ns} and correspond to the strain tensor terms that are related to the damage. If the failure initiates in pure Mode I, then the failure process starts at point A; if the failure initiates in pure Mode II, then the process starts at point B and so on.

[3] This equation is selected for having the most compact expression.

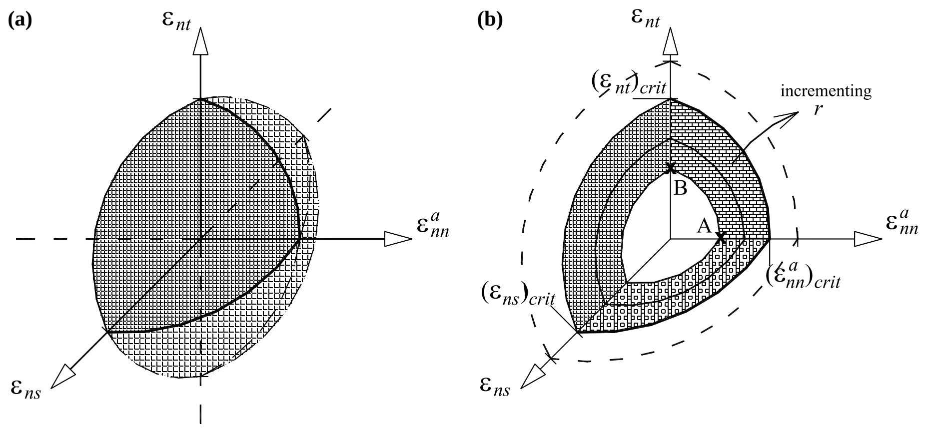

Fig. 6. Damage criterion for the continuum approximation: (a) failure surface and (b) damage evolution.

For the case of failure starting in pure Mode I, the internal variables would take the values: $r^{\mathrm{wd}}[\frac{f_t}{E}, +\infty)$ and $q^{\mathrm{wd}} \in [0, f_t]$, with $q_0^{\mathrm{wd}} = f_t$. The damage variable, calculated by means of (11): $d^{\mathrm{wd}} = 1 - \frac{q^{\mathrm{wd}}(r^{\mathrm{wd}})}{r^{\mathrm{wd}}} \frac{1}{E}$, has a range: $d^{\mathrm{wd}} \in [0, 1]$.

The tangent constitutive operator is obtained by time deriving the constitutive equation

$$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{C}}_S : \boldsymbol{\epsilon} + \mathbf{C}_S : \dot{\boldsymbol{\epsilon}} = d' \dot{r}^{\mathrm{wd}} \mathbf{C}'_S : \boldsymbol{\epsilon} + \mathbf{C}_S : \dot{\boldsymbol{\epsilon}} \tag{51}$$

with

$$\mathbf{C}'_S = \partial_d \mathbf{C}_S \quad \text{and} \quad d' = \partial_r d^{\mathrm{wd}} = \left[(1 - d^{\mathrm{wd}}) - \frac{\mathscr{H}^{\mathrm{wd}}}{E} \right] \frac{1}{r^{\mathrm{wd}}} \tag{52}$$

Considering the Eq. (49), $\dot{r}^{\mathrm{wd}}$ can be calculated as

$$\dot{r}^{\mathrm{wd}} = \dot{\tau}_\epsilon = \frac{1}{\tau_\epsilon} [\epsilon^a_{nn} \mathbf{n} \otimes \mathbf{n} : \mathbf{W}_\epsilon : \mathbf{n} \otimes \mathbf{n} \dot{\epsilon}^a_{nn} + \boldsymbol{\epsilon} : \mathbf{H}^{\circledast} : \mathbf{W}_\epsilon : \mathbf{H}^{\circledast} : \dot{\boldsymbol{\epsilon}}] \tag{53}$$

with

$$\dot{\epsilon}^a_{nn} = \frac{d' \dot{r}^{\mathrm{wd}}}{(1 - d^{\mathrm{wd}})^2 E} \mathbf{n} \otimes \mathbf{n} : \mathbf{C}_S : \boldsymbol{\epsilon} + \frac{d' \dot{r}^{\mathrm{wd}}}{(1 - d^{\mathrm{wd}}) E} \mathbf{n} \otimes \mathbf{n} : \mathbf{C}'_S : \boldsymbol{\epsilon} + \frac{1}{(1 - d^{\mathrm{wd}}) E} \mathbf{n} \otimes \mathbf{n} : \mathbf{C}_S : \dot{\epsilon} \tag{54}$$

and

$$\dot{d}^{\mathrm{wd}} = d' \dot{r}^{\mathrm{wd}} = d' \dot{\tau}_\epsilon \tag{55}$$

To simplify the previous equations let define the constants A_1 and A_3

$$A_1 = \mathbf{n} \otimes \mathbf{n} : \mathbf{W}_\epsilon : \mathbf{n} \otimes \mathbf{n} \epsilon^a_{nn} = (W_\epsilon)_{nnnn} \epsilon^a_{nn} \tag{56}$$

$$A_3 = \frac{d'}{(1 - d^{\mathrm{wd}})^2 E} \mathbf{n} \otimes \mathbf{n} : \mathbf{C}_S : \boldsymbol{\epsilon} + \frac{d'}{(1 - d^{\mathrm{wd}}) E} \mathbf{n} \otimes \mathbf{n} : \mathbf{C}'_S : \boldsymbol{\epsilon} \tag{57}$$

Substituting A_1 and A_3 in (53) and rearranging terms, an expression to calculate $\dot{r}^{\mathrm{wd}}$ as a function of the strain rate $\dot{\boldsymbol{\epsilon}}$ is obtained

$$\dot{r}^{\mathrm{wd}} = \dot{\tau}_{\epsilon} = \frac{1}{(\tau_{\epsilon} - A_1 A_3)} \left[A_1 \left(\frac{1}{(1 - d^{\mathrm{wd}})E} \right) \mathbf{n} \otimes \mathbf{n} : \mathbf{C}_S : \dot{\boldsymbol{\epsilon}} + \boldsymbol{\epsilon} : \mathbf{H}^{\circledast} : \mathbf{W}_{\epsilon} : \mathbf{H}^{\circledast} : \dot{\boldsymbol{\epsilon}} \right] \tag{58}$$

From (51), (52) and (58), the tangent constitutive tensor that relates the stress rates $\dot{\boldsymbol{\sigma}}$ and the strain rates $\dot{\boldsymbol{\epsilon}}$ can be defined:

- For the elastic loading and unloading range ($\dot{d}^{\mathrm{wd}} = 0$)

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}_S : \dot{\boldsymbol{\epsilon}} \tag{59}$$

- For the inelastic and neutral loading range ($\dot{d}^{\mathrm{wd}} > 0$)

$$\dot{\boldsymbol{\sigma}} = \left(\mathbf{C}_S + \left[(1 - d^{\mathrm{wd}}) - \frac{\mathscr{H}^{\mathrm{wd}}}{E} \right] \left[\frac{1}{r^{\mathrm{wd}}} \right] \left[\frac{1}{\tau_{\epsilon} - A_1 A_3} \right] \left[A_1 \left(\frac{1}{(1 - d^{\mathrm{wd}})E} \right) (\mathbf{C}'_S : \boldsymbol{\epsilon}) \otimes (\mathbf{n} \otimes \mathbf{n} : \mathbf{C}_S) + (\mathbf{C}'_S : \boldsymbol{\epsilon}) \otimes (\boldsymbol{\epsilon} : \mathbf{H}^{\circledast} : \mathbf{W}_{\epsilon}) \right] \right) : \dot{\boldsymbol{\epsilon}} \tag{60}$$

Remark 8. Eqs. (59) and (60) are only valid for $[[\epsilon]]_{nn} \geqslant 0$. This is due to the physical sense of the Mode I of fracture; there are just two options: the crack is opened [4] ($[[\epsilon]]_{nn} > 0 \Rightarrow [[u]]_n > 0$) or is closed ($[[\epsilon]]_{nn} > 0 \Rightarrow [[u]]_n > 0$). On the other hand, the values of $[[\epsilon]]_{nt}$ and $[[\epsilon]]_{ns}$ can be positive or negative depending on the sign of the stress components σ_{nt} and σ_{ns}, respectively.

If the stress component σ_{nn} is negative (compression), then it will be necessary to substitute the value of the constitutive tensor term $([C_S]^{-1})_{nnnn} = \frac{1}{(1-d^{\mathrm{wd}})E}$ (Eq. (36)) by the value $([C_S]^{-1})_{nnnn} = \frac{1}{E}$. Then the same procedure and equations established in this section may be used in this model.

Summarizing. In this anisotropic damage model for continuum approximation of discontinuities, the independent variable is only the strain $\boldsymbol{\epsilon}$. The dependent variable is the stress $\boldsymbol{\sigma}$. In this model the damage is localized at Ω_k and there is no damage at $\Omega \setminus \Omega_k$. f_t, $\mathscr{H}^{\mathrm{wd}}$ and k are considered material properties.

6. Energy analysis

6.1. Fracture energy

It is important to develop equations that relate the parameters of the two constitutive models developed here, in such a way that these can be compared. For this purpose, an "Energy Analysis" of a body with a discontinuity is proposed to find those equations. From this energy analysis relationships for the internal variables and the softening modulus are obtained.

The starting point of the energy analysis is to calculate the strain energy of a body with a discontinuity; this is done for both approximations. In this paper, it will be considered that the body behavior is elastic, except for the discontinuity where the damage takes place. The strain energy of the body U considering a discrete approximation can be calculated as

[4] In the continuum approximation there is no crack, there is a localization zone. The term crack is used to facilitate the presentation. To demonstrate that $[[\epsilon]]_{nn} < 0$ is not valid, consider a very small localization zone width ($k \to 0$); if $[[\epsilon]]_{nn} < 0$, then $k < 0$ and $\Omega_k < 0$. But areas might not be negative ($\Omega_k < 0$).

$$U^{\mathrm{d}} = \int_{\Omega\backslash S}\int_0^{\hat{t}_i} \boldsymbol{\sigma} : \dot{\bar{\boldsymbol{\epsilon}}}\,\mathrm{d}\hat{t}\,\mathrm{d}\Omega + \int_S\int_0^{\hat{t}_i} \mathbf{T}_S \cdot [[\dot{\mathbf{u}}]]\,\mathrm{d}\hat{t}\,\mathrm{d}\Gamma \tag{61}$$

where $\hat{t}_i$ is the time in the instant i. The first term corresponds to the energy of the continuous part of the body and is the elastic strain energy. The second one corresponds to the energy associated to the discontinuity.

The strain energy of the body U considering a continuum approximation can be calculated as

$$U^{\mathrm{wd}} = \int_{\Omega}\int_0^{\hat{t}_i} \boldsymbol{\sigma} : \dot{\bar{\boldsymbol{\epsilon}}}\,\mathrm{d}\hat{t}\,\mathrm{d}\Omega + \int_{\Omega_k}\int_0^{\hat{t}_i} \boldsymbol{\sigma} : [[\dot{\boldsymbol{\epsilon}}]]\,\mathrm{d}\hat{t}\,\mathrm{d}\Omega \tag{62}$$

If the localization zone width is considered to be small, then the first terms of (61) and (62) may be considered as equal (Remark 1); these terms correspond to the strain energy associated to the undamaged part of the continuum. If the strain energy of both models is the same, then the following equation must be fulfilled:

$$\int_S\int_0^{\hat{t}_i} \mathbf{T}_S \cdot [[\dot{\mathbf{u}}]]\,\mathrm{d}\hat{t}\,\mathrm{d}\Gamma = \int_{\Omega_k}\int_0^{\hat{t}_i} \boldsymbol{\sigma} : [[\dot{\boldsymbol{\epsilon}}]]\,\mathrm{d}\hat{t}\,\mathrm{d}\Omega \tag{63}$$

Remark 9. The interior integral of the first term of Eq. (63) ($\int_0^{\hat{t}_i} \mathbf{T}_S \cdot [[\dot{\mathbf{u}}]]\,\mathrm{d}\hat{t}$) corresponds to the fracture energy G_{f} of the cohesive crack model of Hillerborg et al. (1976). The interior integral of the second term ($\int_0^{\hat{t}_i} \boldsymbol{\sigma} : [[\dot{\boldsymbol{\epsilon}}]]\,\mathrm{d}\hat{t}$) corresponds to the density of fracture energy γ_{f} of the band crack model of Bažant and Oh (1983). These models are related through the localization zone width k, in such a way that $G_{\mathrm{f}} = k\gamma_{\mathrm{f}}$.

6.2. Relationships between variables

Lets consider the hypostasis that the localization zone width is small, in such a way that the following relationship between the tractions of the discrete approximation and the stresses of the continuum approximation may be established: $(T_S)_n = (\sigma_k)_{nn}$, $(T_S)_t = (\sigma_k)_{nt}$ and $(T_S)_s = (\sigma_k)_{ns}$. These tractions and stresses can be calculated from the displacement jump and the strain jump, as established by the constitutive equations

- Discrete approximation

$$[[u]]_n = \frac{1}{(1-d^{\mathrm{d}})E}T_n \quad [[u]]_t = \frac{1}{(1-d^{\mathrm{d}})G}T_t \quad [[u]]_s = \frac{1}{(1-d^{\mathrm{d}})G}T_s \tag{64}$$

- Continuum approximation

$$[[\epsilon]]_{nn} = \frac{d^{\mathrm{wd}}}{(1-d^{\mathrm{wd}})E}\sigma_{nn} \quad [[\epsilon]]_{nt} = \frac{d^{\mathrm{wd}}}{(1-d^{\mathrm{wd}})(2G)}\sigma_{nt} \quad [[\epsilon]]_{ns} = \frac{d^{\mathrm{wd}}}{(1-d^{\mathrm{wd}})(2G)}\sigma_{ns} \tag{65}$$

Substituting (64) and (65) in (63) (for the two dimensional case)

$$\int_S\int_0^{\hat{t}_i} (1-d^{\mathrm{d}})(E[[u]]_n[[\dot{u}]]_n + G[[u]]_t[[\dot{u}]]_t)\,\mathrm{d}\hat{t}\,\mathrm{d}S$$
$$= \int_{\Omega_k}\int_0^{\hat{t}_i} \frac{(1-d^{\mathrm{wd}})}{d^{\mathrm{wd}}}(E[[\epsilon]]_{nn}[[\dot{\epsilon}]]_{nn} + 4G[[\epsilon]]_{nt}[[\dot{\epsilon}]]_{nt})\,\mathrm{d}\hat{t}\,\mathrm{d}\Omega \tag{66}$$

The kinematics of the continuum approximation establishes that (7): $[[\epsilon]]_{nn} = \frac{1}{k}[[u]]_n$, $[[\epsilon]]_{nt} = \frac{1}{2k}[[u]]_t$ and $[[\epsilon]]_{ns} = \frac{1}{2k}[[u]]_s$. Therefore, to have equal strain energy in both approximations either of the following equations must be fulfilled:

$$(1 - d^{\mathrm{d}}) = \frac{(1 - d^{\mathrm{wd}})}{d^{\mathrm{wd}}}\frac{1}{k} \tag{67}$$

$$\frac{q^{\mathrm{d}}}{Er^{\mathrm{d}}} = \frac{q^{\mathrm{wd}}}{Er^{\mathrm{wd}} - q^{\mathrm{wd}}}\frac{1}{k} \tag{68}$$

To verify the fulfillment of these equations, the relationships between the variables of these models (r^{d} and r^{wd}; q^{d} and q^{wd}) should be established. From the definition of the norm τ_ϵ, r^{wd} can be written as: $r^{\mathrm{wd}} = \frac{1}{d^{\mathrm{wd}}}\sqrt{(W_\epsilon)_{nnnn}([[\epsilon]]_{nn})^2 + 2(W_\epsilon)_{ntnt}([[\epsilon]]_{nt})^2}$. Considering the kinematics of the strain jumps, this equation can be written as: $r^{\mathrm{wd}} = \frac{1}{d^{\mathrm{wd}}}\frac{1}{k}\sqrt{(W_\epsilon)_{nnnn}([[u]]_n)^2 + \frac{1}{2}(W_\epsilon)_{ntnt}([[u]]_t)^2}$. From this equation and given that $(W_\epsilon)_{nnnn} = (W_{[[\mathbf{u}]]})_{nn}$ and $\frac{1}{2}(W_\epsilon)_{ntnt} = (W_{[[\mathbf{u}]]})_{nt}$, the internal variables r^{d} and r^{wd} are related as

$$r^{\mathrm{wd}} = \frac{1}{k}\frac{1}{d^{\mathrm{wd}}}r^{\mathrm{d}} \tag{69}$$

From (69) can be deduced the following equation: $Er^{\mathrm{d}} = E(kd^{\mathrm{wd}}r^{\mathrm{wd}}) = Ek(1 - \frac{1}{E}\frac{q^{\mathrm{wd}}}{r^{\mathrm{wd}}})r^{\mathrm{wd}} = (Er^{\mathrm{wd}} - q^{\mathrm{wd}})k$ and given that $q^{\mathrm{wd}} = q^{\mathrm{d}}$, it may be concluded that Eq. (68) is fulfilled. If (68) is fulfilled, then the discrete and the continuum models are energetically equivalent.

The relationship between the softening modulus of both models is obtained below. This relationship allows to compare the softening curve of these models. Given that $q^{\mathrm{wd}} = q^{\mathrm{d}}$ and if a linear softening modulus is considered (weak discontinuities: $q^{\mathrm{wd}} = q_0 + \mathscr{H}^{\mathrm{wd}}(r^{\mathrm{wd}} - r_0^{\mathrm{wd}})$; Discrete approximation: $q^{\mathrm{d}} = q_0 + \mathscr{H}^{\mathrm{d}}r^{\mathrm{d}}$) the following equation can be established:

$$\frac{\mathscr{H}^{\mathrm{wd}}}{\mathscr{H}^{\mathrm{d}}} = \frac{r^{\mathrm{d}}}{r^{\mathrm{wd}} - r_0^{\mathrm{wd}}} \tag{70}$$

Substituting r^{d}, from Eq. (69): ($r^{\mathrm{d}} = kd^{\mathrm{wd}}r^{\mathrm{wd}}$), in (70) and after some algebraic manipulations a relationship between the softening modulus is obtained

$$\mathscr{H}^{\mathrm{d}} = \frac{\mathscr{H}^{\mathrm{wd}}}{k(1 - \frac{\mathscr{H}^{\mathrm{wd}}}{E})} \tag{71}$$

$$\mathscr{H}^{\mathrm{wd}} = \frac{k\mathscr{H}^{\mathrm{d}}}{1 + k\frac{\mathscr{H}^{\mathrm{d}}}{E}} \tag{72}$$

It should be pointed out that (71) and (72) are also valid for multilinear softening curves [5].

Remark 10. Given a bounded value of $\mathscr{H}^{\mathrm{d}}$, if $k \to 0$ then $\mathscr{H}^{\mathrm{wd}} \to k\mathscr{H}^{\mathrm{d}} \to 0$. This proves that in strong discontinuities the softening modulus is zero: $\mathscr{H}^{\mathrm{wd}} = 0$ (Simo et al., 1993).

[5] The demonstration is straightforward.

7. Conclusions

This paper is focused on the mathematical modeling of discontinuities. It is considered that the discontinuities are caused by localized damage and they are studied with two different approaches: discrete approximation and continuum approximation with weak discontinuities. In the first approximation the constitutive behavior of discontinuities is modeled by ''displacement jump-traction'' relationships and the second one by ''strain–stress'' relationships. The kinematics and the constitutive modeling of discontinuities is presented. From the contents of this paper it can be concluded

- The definition of the kinematics of discontinuities allows to establish the characteristics that the constitutive models must fulfilled. In particular, the kinematics condition for weak discontinuities impose limitations to the direct application of the isotropic damage model, in such a way that it was necessary to propose an Anisotropic one.
- The proposed isotropic damage model for the discrete approximation and the anisotropic damage model for the continuum approximation of weak discontinuities are suitable to model discontinuities. The suitability of the anisotropic damage model comes from the fulfillment of the null tractions and kinematics conditions for discontinuities. Both models have been numerically implemented and used to simulate the fracture of concrete specimens, giving good results. One important characteristic of these models is that they weigh the mode of failure in the failure criterion; this gives a great deal of flexibility in their application. For particular applications, the next step in the development of these models is to define a different damage variable for each mode of fracture; but this falls out of the scope of this paper.
- The energy analysis is a way to guarantee the equivalence between models. This analysis sets the equations that must be fulfilled to make both models equivalent. In addition, the obtained equations allow to relate the discrete approximation model to the fictitious crack model of Hillerborg et al. (1976) by means of the fracture energy and the continuum approximation model to the crack band model of Baẑant and Oh (1983) by means of the density of fracture energy.

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Appendix A. Anisotropic model. Constitutive matrices

In what follows, the anisotropic damage model constitutive matrices are presented. The matrices are defined in a local coordinate system (n, s, t) and they relate the stress vector $(\sigma = \{\sigma_{nn}, \sigma_{tt}, \sigma_{nt}\}^{\mathrm{T}})$ with the strain vector $(\boldsymbol{\epsilon} = \{\epsilon_{nn}, \epsilon_{tt}, \gamma_{nt}\}^{\mathrm{T}})$: $\boldsymbol{\sigma} = \mathbf{C}_S\boldsymbol{\epsilon}$.

- Plane stress $(\boldsymbol{\sigma}_{ss} = 0)$

$$\mathbf{C}_S = \begin{bmatrix} \dfrac{(1-d)E}{(1-v^2+v^2d)} & \dfrac{(1-d)vE}{(1-v^2+v^2d)} & 0 \\ \dfrac{(1-d)vE}{(1-v^2+v^2d)} & \dfrac{E}{(1-v^2+v^2d)} & 0 \\ 0 & 0 & (1-d)G \end{bmatrix} \tag{A.1}$$

- Plane strain ($\epsilon_{ss} = 0$)

$$\mathbf{C}_S = \begin{bmatrix} \frac{(1-d)(1-v)E}{(1-v-2v^2+2v^2d)} & \frac{(1-d)vE}{(1-v-2v^2+2v^2d)} & 0 \\ \frac{(1-d)vE}{(1-v-2v^2+2v^2d)} & \frac{(1-v^2+v^2d)E}{(1-3v^2+2v^2d-2v^3+2v^3d)} & 0 \\ 0 & 0 & (1-d)G \end{bmatrix} \quad \text{(A.2)}$$

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